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EVALUATION OF ELLIPTIC FUNCTIONS OF THE SECOND AND THIRD SPECIES, ANALYST, VOL. V, PAGES 18 & 19.

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4. FOR the second species and ascending scale of moduli we assume the fundamental form:

$$\int_0^\phi d\varphi\sqrt{[a^2-c^2\sin^2\varphi]}=\int_0^\phi a\varDelta(\varphi)d\varphi=aE_0^\phi\left(\frac{c}{a}\right). \tag{21}$$

From (11) follows:

$$\tan\varphi=\frac{a\sin 2\varphi'}{c+a\cos 2\varphi'};$$

$$\therefore\ \sin\varphi=\frac{a\sin 2\varphi'}{\sqrt{[c^2+2ac\cos 2\varphi'+a^2]}}$$

$$=\frac{a\sin\varphi'\cos\varphi'}{\sqrt{[a'^2-c'^2\sin^2\varphi']}}$$

$$=\frac{a\sin\varphi'\cos\varphi'}{a'\varDelta'(\varphi')}, \tag{22}$$

$$a\varDelta(\varphi)=\frac{a}{a'\varDelta'(\varphi')}\sqrt{\left(a'^2-c'^2\sin^2\varphi'-c^2\sin^2\varphi'\cos^2\varphi'\right)}$$

$$=\frac{a}{a'\varDelta'(\varphi')}\left(a'-c\sin^2\varphi'\right)$$

$$=a'\varDelta'(\varphi')+\frac{\frac{1}{4}(a^2-c^2)}{a'\varDelta'(\varphi')}, \tag{23}$$

consequently regarding (12)

$$a\varDelta(\varphi)d\varphi=a^2\varDelta(\varphi)^2.\frac{d\varphi}{a\varDelta(\varphi)}$$

$$=\left[a'\varDelta'(\varphi')+\frac{\frac{1}{4}(a^2-c^2)}{a'\varDelta'(\varphi')}\right]^2\frac{d\varphi'}{a'\varDelta'(\varphi')}$$

$$=a'\varDelta'(\varphi')d\varphi'+\tfrac{1}{2}(a^2-c^2)\frac{d\varphi'}{a'\varDelta'(\varphi')}+\tfrac{1}{16}(a^2-c^2)^2\frac{d\varphi'}{a'^3\varDelta'(\varphi')^3} \tag{24}$$

Differentiating (22) we have

$$a\varphi\cos\varphi = \frac{ad\varphi'}{a'^3\Delta'(\varphi')^3}\left[a'^2(1-2\sin^2\varphi')+c'^2\sin^4\varphi'\right]$$

$$= \frac{ad\varphi'}{a'^3\Delta'(\varphi')^3}\left[\frac{a'^4\Delta'(\varphi')^4}{c'^2}+a'^2-\frac{a'^4}{c'^2}\right]$$

$$= \frac{d\varphi'}{ca'^3\Delta'(\varphi')^3}\left[a'^4\Delta'(\varphi')^4-\tfrac{1}{16}(a^2-c^2)^2\right], \tag{25}$$

$$\therefore\ \tfrac{1}{16}(a^2-c^2)^2\frac{d\varphi'}{a'^3\Delta'(\varphi')^3} = a'\Delta'(\varphi')d\varphi'-c\cos\varphi d\varphi. \tag{25'}$$

Adding this to (24) and integrating we obtain

$$\int_0^\phi a\Delta(\varphi)d\varphi = 2\int_0^{\phi'} a'\Delta'(\varphi)d\varphi+a'(a-c)\int_0^{\phi'}\frac{d\varphi}{a'\Delta'(\varphi)}-c\sin\varphi$$

or $$aE_0^\phi\left(\frac{c}{a}\right) = 2a'E_0^{\phi'}\left(\frac{c'}{a'}\right)+a'(a-c).\frac{1}{a'}F_0^{\phi'}\left(\frac{c'}{a'}\right)-c\sin\varphi. \tag{26}$$

At the next higher step in the scale of moduli we have

$$2a'E_0^{\phi'}\left(\frac{c'}{a'}\right) = 2^2a''E_0^{\phi''}\left(\frac{c''}{a''}\right)+2a''(a'-c').\frac{1}{a''}F_0^{\phi''}\left(\frac{c''}{a''}\right)-2c'\sin\varphi'. \tag{26'}$$

Finally when

$$a^{(n)} = c^{(n)} = \left\|\tfrac{1}{2}\left\{a \underset{\times}{+} c\right\}^{\frac{1}{2}}\right\| \quad \text{we have}$$

$$2^na^{(n)}E_0^{\phi(n)}(1) = 2^na^n\sin\varphi^{(n)}. \tag{26^n}$$

Adding (26), (26′), ... (26$^{(n)}$) we have regarding (13)

$$aE_0^\phi\left(\frac{c}{a}\right) = 2^na^{(n)}\sin\varphi^{(n)}+\Sigma_0^{n-1}[2^ra^{(r+1)}(a^{(r)}-c^{(r)})].\frac{1}{a^{(n)}}\log\tan\tfrac{1}{2}(\tfrac{1}{2}\pi+\varphi^{(n)})$$
$$-\Sigma_0^{n-1}[2^rc^{(r)}\sin\varphi^{(r)}]. \tag{27}$$

This is the length of an arc of an ellipse the equation of which is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1.$$

between $x = 0$ and $x = a\sin\varphi$.

5. For the second species and descending scale we take the form:

$$\int_0^\phi d\varphi\sqrt{[a^2\cos^2\varphi+b^2\sin^2\varphi]} = \int_0^\phi a\Delta(\varphi)d\varphi = aE_0^\phi\sqrt{\left(1-\frac{b^2}{a^2}\right)}. \tag{28}$$

Consistent with (15) we have also

$$\sin(2\varphi-\varphi_1) = \frac{a-b}{a+b}\sin\varphi_1 = \cos\varphi_1(\sin 2\varphi-\tan\varphi_1\cos 2\varphi); \tag{29}$$

for if we add and then subtract the identity

$$\sin\varphi_1 = \sin\varphi_1$$

and divide the difference by the sum (15) results.

Solving for $\cos 2\varphi$ we have

$$\cos 2\varphi = -\frac{a-b}{a+b}\sin^2\varphi_1 \pm \cos\varphi_1 \Delta_1(\varphi_1) \tag{30}$$

If $\Delta_1(\varphi_1) = \sqrt{\{1-[1-(b_1^2 \div a_1^2)]\sin^2\varphi_1\}}$ is always taken of the same sign as $\cos\varphi_1 = \sqrt{[1-\sin^2\varphi_1]}$ only the upper sign can be used; for if $b = 0$ we must have

$$\cos 2\varphi = -\sin^2\varphi_1 + \cos^2\varphi_1 = \cos 2\varphi_1 \text{ or } \varphi = \varphi_1.$$

We have then
$$\sin^2\varphi = \tfrac{1}{2}\left[1+\frac{a-b}{a+b}\sin^2\varphi_1 - \cos\varphi_1 \Delta_1(\varphi_1)\right] \tag{31}$$

and
$$\begin{aligned} a^2\Delta(\varphi)^2 &= a^2-(a^2-b^2)\sin^2\varphi \\ &= a^2-\tfrac{1}{2}(a-b)[a+b+(a-b)\sin^2\varphi_1 - 2a_1\cos\varphi_1\Delta_1(\varphi_1)] \\ &= 2a_1^2\Delta_1(\varphi_1)^2 - ab + (a-b)a_1\cos\varphi_1\Delta_1(\varphi_1), \end{aligned} \tag{32}$$

consequently regarding (16)

$$\begin{aligned} a\Delta(\varphi)d\varphi &= a^2\Delta(\varphi)^2 . \frac{d\varphi}{a\Delta(\varphi)} \\ &= [2a_1^2\Delta_1(\varphi_1)^2 - b_1^2 + (a-b)a_1 \cos\varphi_1\Delta_1(\varphi_1)]\frac{d\varphi_1}{2a_1\Delta_1(\varphi_1)} \\ &= a_1\Delta_1(\varphi_1)d\varphi_1 - \tfrac{1}{2}b_1^2\frac{d\varphi_1}{a_1\Delta_1(\varphi_1)} + \tfrac{1}{2}(a-b)d\varphi_1\cos\varphi_1. \end{aligned} \tag{33}$$

Integrating we have

$$\int_0^\phi a\Delta(\varphi)d\varphi = \int_0^{\phi_1} a_1\Delta_1(\varphi)d\varphi - \tfrac{1}{2}b_1^2\int_0^{\phi_1}\frac{d\varphi}{a_1\Delta_1(\varphi)} + \tfrac{1}{2}(a-b)\sin\varphi_1, \tag{34}$$

or
$$aE_0^\phi\sqrt{\left\{1-\frac{b^2}{a^2}\right\}} = a_1E_0^{\phi_1}\sqrt{\left\{1-\frac{b_1^2}{a_1^2}\right\}} - \tfrac{1}{2}b_1^2 . \frac{1}{a_1}F_0^{\phi_1}\sqrt{\left\{1-\frac{b_1^2}{a_1^2}\right\}} + \tfrac{1}{2}(a-b)\sin\varphi_1. \tag{35}$$

Similarly we have

$$a_1E_0^{\phi_1}\sqrt{\left\{1-\frac{b_1^2}{a_1^2}\right\}} = a_2E_0^{\phi_2}\sqrt{\left\{1-\frac{b_2^2}{a_2^2}\right\}} - \tfrac{1}{2}b_2^2 . \frac{1}{a_2}F_0^{\phi_2}\sqrt{\left\{1-\frac{b_2^2}{a_2^2}\right\}} + \tfrac{1}{2}(a_1-b_1)\sin\varphi_2, \tag{35$_1$}$$

. . . .

$$a_nE_0^{\phi_n}\sqrt{\left\{1-\frac{b_n{}^2}{a_n{}^2}\right\}} = a_nE_0^{\phi_n}(0) = a_n\varphi_n, \text{ where } a_n = b_n = \left\|\tfrac{1}{2}\left(a \overset{+}{\times} b\right)^{\frac{1}{2}}\right\|. \tag{35$_n$}$$

Adding (35), (35_1), . . . (35_n) we obtain regarding (19)

$$\begin{aligned} aE_0^\phi\sqrt{\left\{1-\frac{b^2}{a^2}\right\}} &= b_n\varphi_n - \frac{\varphi_n}{b_n}\left\{\frac{b_n^2}{2}+\frac{b_{n-1}^2}{2^2}+\dots\frac{b_1^2}{2^n}\right\} \\ &\quad + \tfrac{1}{2}(a-b)\sin\varphi_1 + \tfrac{1}{2}(a_1-b_1)\sin\varphi_2 + \dots \tfrac{1}{2}(a_{n-1}-b_{n-1})\sin\varphi_u \\ &= \tfrac{1}{2}b_n\varphi_n - \frac{\varphi_n}{b_n}\Sigma_1^{n-1}\left\{\frac{b_r{}^2}{2^{n-r+1}}\right\} + \tfrac{1}{2}\Sigma_1^n[(a_{r-1}-b_{r-1})\sin\varphi_r]. \end{aligned} \tag{36}$$

If $\varphi = \frac{1}{2}\pi$ then $\varphi_1 = \pi$; $\varphi_2 = 2\pi, .. \varphi_n = 2^{n-1}\pi$ and $\sin\varphi_1 = \sin\varphi_2 = .. \sin\varphi_n = 0$,

hence
$$aE_0^{\frac{\pi}{2}}\sqrt{\left\{1-\frac{b^2}{a^2}\right\}} = 2^{n-2}b_n\pi - \frac{\pi}{b_n}\Sigma_1^{n-1}[2^{r-2}b_r^2]. \tag{36$'$}$$

Formula (34) might have been also derived from (26) by regarding the following relations, which exist between the quantities with primes and those with subscripts if each series is continued in both directions. We have then the amplitudes

$$\varphi^{(n)}=\varphi_{-n};\ldots\varphi'=\varphi_{-1};\ \varphi=\varphi;\ \varphi_{\prime}=\varphi_1;\ \varphi_{\prime\prime}=\varphi_2;\ldots\varphi_{(n)}=\varphi_n; \quad (37)$$

also the moduli

$$\frac{c^{(n)}}{a^{(n)}}=\left\{1-\frac{b^2_{-n}}{a^2_{-n}}\right\}^{\frac{1}{2}};\ldots\frac{c'}{a'}=\left\{1-\frac{b^2_{-1}}{a^2_{-1}}\right\}^{\frac{1}{2}};\ \frac{c}{a}=\left\{1-\frac{b^2}{a^2}\right\}^{\frac{1}{2}};$$

$$\frac{c_{\prime}}{a_{\prime}}=\left\{1-\frac{b_1^2}{a_1^2}\right\}^{\frac{1}{2}};\ldots\frac{c_{(n)}}{a_{(n)}}=\left\{1-\frac{b_n^2}{a_n^2}\right\}^{\frac{1}{2}}; \quad (38)$$

whence

$$\Delta^{(n)}(\varphi)=\Delta_{-n}(\varphi);\ldots\Delta'(\varphi)=\Delta_{-1}(\varphi);\ \Delta\varphi=\Delta\varphi;\ \Delta_{\prime}(\varphi)=\Delta_1(\varphi);$$
$$\ldots\Delta_{(n)}(\varphi)=\Delta_n(\varphi). \quad (39)$$

By (12) we have

$$\int_0^{\phi^{(n)}}\frac{d\varphi}{a^{(n)}\Delta^{(n)}(\varphi)}=\int_0^{\phi^{(n-1)}}\frac{d\varphi}{a^{(n-1)}\Delta^{(n-1)}(\varphi)}=\cdots\int_0^{\phi'}\frac{d\varphi}{a'\Delta'(\varphi)}=\int_0^{\phi}\frac{d\varphi}{a\,\Delta(\varphi)}$$
$$=\int_0^{\psi_{\prime}}\frac{d\psi^*}{a_{\prime}\Delta_{\prime}(\psi)}=\cdots\int_0^{\psi_{(n)}}\frac{d\psi}{a_{(n)}\Delta_{(n)}(\psi)} \quad (40)$$

while by (16) we have

$$\int_0^{\psi_{-n}}\frac{d\psi}{2^{-n}a_{-n}\Delta_{-n}(\psi)}=\int_0^{\psi_{-n+1}}\frac{d\psi}{2^{-n+1}a_{-n+1}\Delta_{-n+1}(\psi)}=\cdots\int_0^{\psi_{-1}}\frac{d\psi}{2^{-1}a_{-1}\Delta_{-1}(\psi)}$$
$$=\int_0^{\psi}\frac{d\psi}{a\Delta\psi}=\int_0^{\psi_1}\frac{d\psi}{2a_1\Delta_1(\psi)}=\cdots\int_0^{\psi_n}\frac{d\psi}{2^na_n\Delta_n(\psi)} \quad (41)$$

We have then regarding (37) and (39)

$$a^{(n)}=2^{-n}a_{-n};\ a^{(n-1)}=2^{-n+1}a_{-n+1};\ldots a'=2^{-1}a_{-1};\ a=a;\ a_{\prime}=2a_1;$$
$$\ldots a_{(n)}=2^na_n \quad (42)$$

These relations enable us to derive (34) from (26) and vice versa.

We have by (26) for the next lower step in the modular scale

$$\int_0^{\psi}a_{\prime}\Delta_{\prime}(\psi)d\psi=2\int_0^{\psi}a\Delta(\psi)d\psi+\tfrac{1}{2}(a_{\prime}^2-c_{\prime}^2)\int_0^{\psi}\frac{d\psi}{a\Delta(\psi)}-c_{\prime}\sin\psi_{\prime}$$

$$\therefore\quad\int_0^{\psi}a\,\Delta(\psi)d\psi=\tfrac{1}{2}\int_0^{\psi_{\prime}}a_{\prime}\Delta_{\prime}(\psi)d\psi-\tfrac{1}{4}(a_{\prime}^2-c_{\prime}^2)\int_0^{\psi}\frac{d\psi}{a\Delta(\psi)}+\tfrac{1}{2}c_{\prime}\sin\psi_{\prime}.$$

But $\psi_{\prime}=\psi_1$; $a_{\prime}=2a_1$; $\Delta_{\prime}(\psi)=\Delta_1(\psi)$; $a_{\prime}^2-c_{\prime}^2=4b_1^2$; $c_{\prime}=2\sqrt{(a_1^2-b_1^2)}$ $=a-b$ and $\displaystyle\int_0^{\psi}\frac{d\psi}{a\Delta(\psi)}=\int_0^{\psi_1}\frac{d\psi}{2a_1\Delta_1(\psi)}$ by (41),

$$\therefore\quad\int_0^{\psi}a\Delta(\psi)d\psi=\int_0^{\psi_1}a_1\Delta_1(\psi)d\psi-\tfrac{1}{2}b_1^2\int_0^{\psi_1}\frac{d\psi}{a_1\Delta_1(\psi)}+\tfrac{1}{2}(a-b)\sin\psi_1,$$

which is (34).

[*For want of sorts, ψ is here, and thrughout the rest of this article, used for ϕ —Comp.]

These relations are also very useful for the inverse problem of determining the amplitude to a given integral. It is not my purpose however to show this now.

6. For the third species and ascending scale we take the fundamental form:

$$\int_0^{\psi}\frac{d\phi}{(p+r\sin^2\phi)\sqrt{[a^2-c^2\sin^2\phi]}}=\int_0^{\psi}\frac{d\phi}{aP(\phi)\triangle(\phi)}=\frac{1}{ap}\Pi_0^{\psi}\left(\frac{c}{a},\frac{r}{p}\right). \tag{43}$$

We have, substituting (22)

$$\frac{1}{P(\psi)}=\frac{a'^2-c'^2\sin^2\psi'}{a'^2p+a(ar-cp)\sin^2\psi'-a^2r\sin^4\psi'} \tag{44}$$

Assume

$$a'^4p^2+aa'^2p(ar-cp)\sin^2\psi'-a^2a'^2pr\sin^4\psi'=(a'^2p+am'\sin^2\psi')(a'^2p+am^{\backprime}\sin^2\psi')$$

whence $\qquad m'+m^{\backprime}=ar-cp;\ m'm^{\backprime}=-a'^2pr \qquad (45)$

and $\qquad m'-m^{\backprime}=\sqrt{[(p+r)(ar^2+cp^2)]}. \qquad (46)$

Now let

$$\frac{\frac{1}{p}-\frac{ac}{a'^2p}\sin^2\psi'}{\left(1+\frac{am'}{a'^2p}\sin^2\psi'\right)\left(1+\frac{am^{\backprime}}{a'^2p}\sin^2\psi'\right)}=\frac{1}{p'+r'\sin^2\psi'}+\frac{1}{p^{\backprime}+r^{\backprime}\sin^2\psi'}$$

$$=\frac{1}{p'}+\frac{1}{p^{\backprime}}+\frac{r'+r^{\backprime}}{p'p^{\backprime}}\sin^2\psi'\div\left(1+\frac{r'}{p'}\sin^2\psi'\right)\left(1+\frac{r^{\backprime}}{p^{\backprime}}\sin^2\psi'\right).$$

By comparison and regarding (45) we have

$$r'=\frac{am'p'}{a'^2p}=-\frac{p'}{m^{\backprime}}ar;\qquad r^{\backprime}=\frac{am^{\backprime}p^{\backprime}}{a'^2p}=-\frac{p^{\backprime}}{m'}ar. \tag{47}$$

$$\frac{1}{p}=\frac{1}{p'}+\frac{1}{p^{\backprime}};\qquad -c=\frac{m'}{p^{\backprime}}+\frac{m^{\backprime}}{p'} \tag{48}$$

whence $$p'=\frac{m'-m^{\backprime}}{m'+cp}p;\qquad p^{\backprime}=-\frac{m'-m^{\backprime}}{m^{\backprime}+cp}p. \tag{49}$$

Thus p', $p^{\backprime}$ and r', $r^{\backprime}$ are known, and placing $P'(\phi)=p'+r'\sin^2\phi$; $P^{\backprime}(\phi)=p^{\backprime}+r^{\backprime}\sin^2\phi$ we have regarding (12) or (41)

$$\int_0^{\psi}\frac{d\phi}{aP(\phi)\triangle(\phi)}=\int_0^{\psi'}\frac{d\phi}{a'P'(\phi)\triangle'(\phi)}+\int_0^{\psi'}\frac{d\phi}{a'P^{\backprime}(\phi)\triangle'(\phi)} \tag{50}$$

At the next higher step in the modular scale we have besides (11′) the formulæ

$$m''+m'^{\backprime}=a'r'-c'p',\qquad m^{\backprime\prime}+m^{\backprime\backprime}=a'r^{\backprime}-c'p^{\backprime}, \tag{45'}$$

$$m''-m'^{\backprime}=\sqrt{[(p'+r')(a'^2r'+c'^2p')]},\quad m^{\backprime\prime}-m^{\backprime\backprime}=\sqrt{[(p^{\backprime}+r^{\backprime})(a'^2r^{\backprime}+c'^2p^{\backprime})]}, \tag{46'}$$

$$p''=\frac{m''-m'^{\backprime}}{m''+c'p'}p';\ p'^{\backprime}=-\frac{m''-m'^{\backprime}}{m'^{\backprime}+c'p'}p'\ \ p^{\backprime\prime}=\frac{m^{\backprime\prime}-m^{\backprime\backprime}}{m^{\backprime\prime}+c'p^{\backprime}}p^{\backprime};\ p^{\backprime\backprime}=-\frac{m^{\backprime\prime}-m^{\backprime\backprime}}{m^{\backprime\backprime}+c'p^{\backprime}}p^{\backprime}$$

$$\ldots (49')$$

$$r'' = -\frac{p''}{m'^{\backprime}}a'r'; \; r'^{\backprime} = -\frac{p'^{\backprime}}{m''}a'r' \quad r^{\backprime\prime} = -\frac{p^{\backprime\prime}}{m^{\backprime\backprime}}a'r^{\backprime}; \; r^{\backprime\backprime} = -\frac{p^{\backprime\backprime}}{m^{\backprime\prime}}a'r^{\backprime}; \quad (47')$$

and if $P''(\psi) = p'' + r''\sin^2\psi$; $P'^{\backprime}(\psi) = p'^{\backprime} + r'^{\backprime}\sin^2\psi$; $P^{\backprime\prime}(\psi) = p^{\backprime\prime} + r^{\backprime\prime}\sin^2\psi$; $P^{\backprime\backprime}(\psi) = p^{\backprime\backprime} + r^{\backprime\backprime}\sin^2\psi$ we have

$$\int_0^{\psi}\frac{d\psi}{aP(\psi)\triangle(\psi)} = \begin{array}{l}\displaystyle\int_0^{\phi'}\frac{d\psi}{a'P'(\psi)\triangle'(\psi)} \\ \displaystyle+\int_0^{\phi'}\frac{d\psi}{a'P^{\backprime}(\psi)\triangle'(\psi)}\end{array} = \begin{array}{l}\displaystyle\int_0^{\phi''}\frac{d\psi}{a''P''(\psi)\triangle''(\psi)} \\ \displaystyle+\int_0^{\phi''}\frac{d\psi}{a''P'^{\backprime}(\psi)\triangle''(\psi)} \\ \displaystyle+\int_0^{\phi''}\frac{d\psi}{a''P^{\backprime\prime}(\psi)\triangle''(\psi)} \\ \displaystyle+\int_0^{\phi''}\frac{d\psi}{a''P^{\backprime\backprime}(\psi)\triangle''(\psi)}\end{array} \quad (51)$$

Pursuing the ascending scale of moduli until $a^{(n)} = c^{(n)} = \left\| \frac{1}{2}\left(a \overset{+}{\times} c\right)^{\frac{1}{2}} \right\|$ and consequently $\triangle^{(n)}(\psi) = \cos\psi$, we have finally 2^n integrals of the form:

$$\int_0^{\phi(n)}\frac{d\psi}{a^{(n)}P^{(n)}(\psi)\cos\psi} = \frac{r^{(n)}}{a^{(n)}(p^{(n)}+r^{(n)})}\int_0^{\phi(n)}\frac{d\psi\cos\psi}{P^{(n)}(\psi)} + \frac{1}{a^{(n)}(p^{(n)}+r^{(n)})}\int_0^{\phi(n)}\frac{d\psi}{\cos\psi}$$
$$= \frac{\sqrt{r^{(n)}}}{a^{(n)}(p^{(n)}+r^{(n)})\sqrt{p^{(n)}}}\tan^{-1}\left\{\sqrt{\left(\frac{r^{(n)}}{p^{(n)}}\right)}\sin\psi^{(n)}\right\}$$
$$+\frac{1}{a^{(n)}(p^{(n)}+r^{(n)})}\log\tan\tfrac{1}{2}(\tfrac{1}{2}\pi+\psi^{(n)}). \quad (51)$$

These reductions are very troublesome if p', $p^{\backprime}$, ... r', $r^{\backprime}$, ... are complex quantities. In this case and if a problem leads to the form

$$\int_0^{\phi} f(\sin^2\psi)\frac{d\psi}{a\triangle(\psi)}$$

(f denoting a rational function of $\sin^2\psi$) which might be expressed by means of some of the three species of elliptic functions and elementary forms, it is perhaps better to express $f(\sin^2\psi)$ successively in terms of $\sin^2\psi'$, $\sin^2\psi''$ ... $\sin^2\psi^{(n)}$ so that if $f'(\sin^2\psi)$, $f''\sin^2\psi)$, ... are the resulting forms we have

$$\int_0^{\phi} f(\sin^2\psi)\frac{d\psi}{a\triangle(\psi)} = \int_0^{\phi'} f'(\sin^2\psi)\frac{d\psi}{a'\triangle'(\psi)} = \int_0^{\phi''} f''(\sin^2\psi)\frac{d\psi}{a''\triangle''(\psi)}\cdots$$
$$= \int_0^{\phi(n)} f^{(n)}(\sin^2\psi)\frac{d\psi}{a^{(n)}\cos\psi} = \frac{1}{a^{(n)}}\int_0^{\sin\phi^{(n)}} f(x^2)\frac{dx}{1-x^2}. \quad (52)$$

7. For the third species and descending scale we take the form:

$$\int_0^{\phi}\frac{d\psi}{(p\cos^2\psi+q\sin^2\psi)\sqrt{[a^2\cos^2\psi+b^2\sin^2\psi]}} = \int_0^{\phi}\frac{d\psi}{aP(\psi)\triangle(\psi)}$$
$$= \Pi_0^{\phi}\left\{\sqrt{\left(1-\frac{b^2}{a^2}\right)}, \frac{p}{q}-1\right\}. \quad (53)$$

Substituting (31) we have regarding (16) or (41)

$$\frac{d\psi}{aP(\psi)\triangle(\psi)} = \frac{1}{p+\frac{1}{2}(q-p)\left[1+\dfrac{a-b}{a+b}\sin^2\psi_1 - \cos\varphi_1\triangle_1(\psi_1)\right]}\,\frac{d\psi_1}{2a_1\triangle_1(\psi)}$$

$$=\frac{p+\frac{1}{2}(q-p)\left[1+\frac{a-b}{a+b}\sin^2\psi_1+\cos^2\psi_1\triangle_1(\psi)\right]}{pq+\frac{q-p}{(a+b)^2}\left(a^2q-b^2p\right)\sin^2\psi_1}\cdot\frac{d\psi_1}{2a_1\triangle_1(\psi_1)}$$

$$=\frac{a^2q^2-b^2p^2}{4(a^2q-b^2p)}\cdot\frac{1}{pq\cos^2\psi_1+\left(\frac{aq+bp}{a+b}\right)^2\sin^2\psi_1}\cdot\frac{d\psi_1}{a_1\triangle_1(\psi_1)}$$

$$+\frac{a^2-b^2}{4(a^2q-b^2p)}\cdot\frac{d\psi_1}{a_1\triangle_1(\psi_1)}+\frac{q-p}{4a_1}\cdot\frac{d\psi_1\cos\psi_1}{pq\cos^2\psi_1+\left(\frac{aq+bp}{a+b}\right)^2\sin^2\psi_1}. \quad (54)$$

Let
$$m_1=\frac{a^2q^2-b^2p^2}{a^2q-b^2p}, \quad (55)$$

$$p_1=\frac{q}{m_1}p, \quad (56)$$

$$q_1=\frac{(aq+bp)^2}{4a_1^2m_1}. \quad (57)$$

We have then if $P_1(\psi_1)=p_1\cos^2\psi_1+q_1\sin^2\psi_1$

$$\int_0^\phi\frac{d\psi}{aP(\psi)\triangle(\psi)}=\frac{1}{4}\left\{\int_0^{\phi_1}\frac{d\psi}{a_1P_1(\psi)\triangle_1(\psi)}+\frac{a^2-b^2}{a^2q-b^2p}\int_0^{\phi_1}\frac{d\psi}{a_1\triangle_1(\psi)}\right.$$
$$\left.+\frac{q-p}{a_1m_1}\int_0^{\phi_1}\frac{d\psi\cos\psi}{P_1(\psi)}\right\};$$

or if we put
$$q-p=r;\ q_1-p_1=r_1 \quad (58)$$

we have $r_1=\dfrac{a^2q-b^2p}{4a_1^2m_1}r$; hence $\dfrac{a^2-b^2}{a^2q-b^2p}=\dfrac{(a-b)r}{2a_1m_1r_1}$ and

$$\int_0^\phi\frac{d\psi}{aP(\psi)\triangle(\psi)}=\frac{1}{4}\left\{\int_0^{\phi_1}\frac{d\psi}{a_1P_1(\psi)\triangle_1(\psi)}+\frac{(a-b)r}{2a_1m_1r_1}\int_0^{\phi_1}\frac{d\psi}{a_1\triangle_1(\psi)}\right.$$
$$\left.+\frac{r}{a_1m_1}\int_0^{\phi_1}\frac{d\psi\cos\psi}{P_1(\psi)}\right\}. \quad (59)$$

For the next lower step in the modular scale we have

$$\frac{1}{4}\int_0^{\phi_1}\frac{d\psi}{a_1P_1(\psi)\triangle_1(\psi)}=\frac{1}{4^2}\left\{\int_0^{\phi_2}\frac{d\psi}{a_2P_2(\psi)\triangle_2(\psi)}+\frac{(a_1-b_1)r_1}{2a_2m_2r_2}\int_0^{\phi_2}\frac{d\psi}{a_2\triangle_2(\psi)}\right.$$
$$\left.+\frac{r_1}{a_2m_2}\int_0^{\phi_2}\frac{d\psi\cos\psi}{P_2(\psi)}\right\}, \quad (59_1)$$

where we have besides (17)

$$m_2=\frac{a_1^2q_1^2-b_1^2p_1^2}{a_1^2q_1-b_1^2p_1}, \quad (55_1)$$

$$p_2=\frac{q_1}{m_2}p_1, \quad (56_1)$$

$$q_2 = \frac{(a_1 q_1 + b_1 p_1)^2}{4a_2^2 m_2}, \tag{57_1}$$

$$r_2 = \frac{a_1^2 q_1 - b_1^2 p_1}{4a_2^2 m_2} r_1 = q_2 - p_2. \tag{58_1}$$

Descending the modular scale until $a_n = b_n = \| \frac{1}{2}(a \overset{+}{\times} b)^{\frac{1}{2}} \|$ we have finally

$$\frac{1}{4^n}\int_0^{\psi_n} \frac{d\psi}{a_n P_n(\psi)} = \frac{1}{4^n a_n \sqrt{(p_n q_n)}} \tan^{-1}\left\{ \sqrt{\left(\frac{q_n}{p_n}\right)} \tan \psi_n \right\}. \tag{59_n}$$

Adding (59), (59_1), ... (59_n) we obtain

$$\int_0^{\psi} \frac{d\psi}{aP(\psi)\triangle(\psi)} = \frac{1}{4^n a_n \sqrt{(p_n q_n)}} \tan^{-1}\left\{ \left(\frac{q_n}{p_n}\right)^{\frac{1}{2}} \tan \psi_n \right\} + \sum_1^n \left\{ \frac{(a_{s-1} - b_{s-1}) r_{s-1}}{2^{2s+1} a_s m_s r_s} \right.$$

$$\left. \times \int_0^{\psi_s} \frac{d\psi}{a_s \triangle_s(\psi)} \right\} + \sum_1^n \left\{ \frac{r_{s-1}}{2^{2s} a_s m_s} \int_0^{\psi_s} \frac{d\psi \cos \psi}{P_s(\psi)} \right\}. \tag{60}$$

But
$$\int_0^{\psi_s} \frac{d\psi}{a_s \triangle_s(\varphi)} = \frac{\psi_n}{2^{n-s} a_n} \tag{61}$$

and
$$\int_0^{\psi_s} \frac{d\psi \cos \psi}{P_s(\psi)} = \frac{1}{\sqrt{(p_s r_s)}} \tan^{-1}\left\{ \left(\frac{r_s}{p_s}\right)^{\frac{1}{2}} \sin \psi_s \right\}, \tag{62}$$

consequently

$$\int_0^{\psi} \frac{d\psi}{aP(\psi)\triangle(\psi)} = \frac{\tan^{-1}[\sqrt{(q_n \div p_n)} \tan \psi_n]}{2^{2n} a_n \sqrt{(p_n q_n)}} + \sum_1^n \left\{ \frac{(a_{s-1} - b_{s-1}) r_{s-1}}{2^{n+s+1} a_s m_s r_s} \right\} \frac{\psi_n}{a_n}$$

$$+ \sum_1^n \left\{ \frac{r_{s-1}}{2^{2s} a_s m_s \sqrt{(p_s r_s)}} \tan^{-1}\left[\left(\frac{r_s}{p_s}\right)^{\frac{1}{2}} \sin \psi_s \right] \right\}. \tag{63}$$

It is hardly necessary to state that if p_s has the opposite sign to q_s or r_s, the $\tan^{-1}$ must be replaced by logarithms.

If $\psi = \frac{1}{2}\pi$ then $\psi_1 = \pi$; $\psi_2 = 2\pi$; ... $\psi_n = 2^{n-1}\pi$; hence

$$\int_0^{\frac{\pi}{2}} \frac{d\psi}{aP(\psi)\triangle(\psi)} = \frac{\pi}{2^{n+1} a_n \sqrt{(p_n q_n)}} + \sum_1^n \left\{ \frac{(a_{s-1} - b_{s-1}) r_{s-1}}{2^{s+2} a_s m_s r_s} \right\} \frac{\pi}{a_n}, \tag{64}$$

for

$$\int_0^{2^{s-1}\pi} \frac{d\psi \cos \psi}{P_s(\psi)} = 0 = \int_0^{\pi} \frac{d\psi \cos \psi}{P_s(\varphi)},$$

because the elements of this integral from 0 to $\frac{1}{2}\pi$ are equal and opposite in sign to those from $\frac{1}{2}\pi$ to π.

Equation (59) could be inverted by means of the relations (38), ... (42). I have not yet succeeded however in reducing the relation between the p, q and r to a convenient form. The reduction by ascending the modular scale as made under No. 6 seems to be preferable.